LINEARIZING QUADRATIC TRANSFORMATIONS IN GENETIC ALGEBRAS

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1. Introduction

In the deterministic theory of randomly mating infinite populations, in which there is no differential selection or fertility, certain types of quadratic transformations connecting one generation with the succeeding one are susceptible of a complete mathematical treatment because of their inherently simple structure. A quadratic transformation of an algebra is one which involves quadratic functions of the coordinates. The technique used to study these transformations was first introduced by Haldane [**9**] in a genetical context in 1930 in connection with polyploids and is a method of linearizing the transformations by extending the original vector space sufficiently, using functions of the coordinates, until the transformation becomes linear. The process is described as *linearization* and was studied in more detail, by Bennett [**2**] in connection with linked loci, and in an algebraical context by Holgate [**11**]. Holgate's paper gives a useful brief introduction to genetic algebras.

In this paper we study some questions which arise directly from Holgate's paper and obtain some more explicit results.

We shall use the abbreviation GA for a genetic algebra as defined by Schafer [**17**], for which we shall use the canonical form given by Gonshor [**8**]. Let A_n denote the general GA of dimension $n+1$ with *canonical basis* $c_0, c_1, \ldots, c_n$. We shall assume for simplicity that c_0 is an idempotent element. Algebras arising in practice usually possess idempotents; Gonshor has given conditions for their existence [**7**]. Then multiplication is defined by

$$c_i c_j = c_j c_i = \sum_{k=0}^{n} \lambda_{ijk} c_k,$$

where

$$\lambda_{000} = 1, \quad \lambda_{0jk} = 0 \text{ for } k < j, \quad \lambda_{00k} = 0 \text{ for } k > 0,$$

$$\lambda_{ijk} = 0 \text{ for } (i,j) > 0 \text{ and } k \leqslant \max(i,j).$$

λ_{0jj} are called the *train roots* of A_n. Those of the λ_{ijk} *which are not zero by definition* will be called the constants, or structure constants of the algebra. A_n is a commutative but usually non-associative algebra. It is a *baric algebra* [5]. The baric property implies that if x is a general element representing a population, it can be written uniquely as

$$x = c_0 + x_1c_1 + x_2c_2 + \ldots + x_nc_n,$$

where the coefficient of c_0 is 1. We say that x is of *unit weight*, the *weight* of an element in general being the coefficient of c_0. A *special train algebra* [5, 7] can be defined as a GA in which all powers of the ideal with basis $\{c_1, \ldots, c_n\}$ (the *nilideal*) are ideals.

The *plenary powers* representing random mating between discrete non-overlapping generations are $x, x^{[2]}, x^{[3]}, \ldots, x^{[n]}, \ldots$, where

$$x^{[n]} = x^{[n-1]}x^{[n-1]},$$

and derive from the successive application of the quadratic transformation $\varphi\colon x \to x^2$ which in general will be a quadratic function of $x_1, x_2, \ldots, x_n$.

B_n will denote the corresponding linearized vector space (with respect to φ and x) which will have a basis defined by coordinates which are monomial functions $x_1^{k_1}x_2^{k_2}\ldots x_n^{k_n}$ (where the k_i are non-negative integers). These will be called the *linearizing* (*coordinate*) *functions*. This technique is subsequently defined and discussed very fully. The present problem is two-fold:

(i) to find the dimension of B_n exactly, recursively, or asymptotically; and

(ii) to describe precisely the monomials required or, equivalently, to generate them explicitly.

We shall work in terms of a fixed canonical basis. However, as is well known, such bases are not unique—although the train roots are invariant. We shall show that the dimension of B_n is independent of the basis in A_n. Haldane, working in a non-algebraic context, used the natural genetic basis and it is intuitively evident that the dimension of B_n is an invariant of A_n and the particular quadratic transformation under consideration. However, the linearizing functions are not characterized uniquely. With respect to a canonical basis in A_n they can be taken as monomials, but for a general basis in A_n they will usually be homogeneous polynomial functions in n variables.

The origins of this paper are interesting and have three contemporaneous sources. Initially it was a remark by Holgate on finding an asymptotic value for $\dim B_n$ in the general case. Independently I was looking at the genetic algebra of polyploids with several linked loci. It seemed to me that the techniques necessary were an extension of the case for the genetic algebra of diploids with several linked loci but treated in a rather different way than had hitherto been attempted. This was by extending a technique of Etherington [5] and elucidating more fully the explicit nature of the plenary roots and in particular the question of multiplicities. This in turn depended on a thorough and explicit understanding of the linearization technique. We study in a subsequent paper the induced linear transformation in B_n and the plenary roots of the algebra with respect to quadratic transformations representing random mating.

2. Linearization

To illustrate some features of the linearization technique which Holgate developed [11] let us consider the following example discussed by him.

EXAMPLE 1. Consider the algebra of tetraploidy A_2.

	c_0	c_1	c_2
c_0	c_0	$\frac{1}{2}c_1$	$\frac{1}{6}c_2$
c_1		$\frac{1}{6}c_2$	0
c_2			0

$$x = c_0 + x_1 c_1 + x_2 c_2.$$

The quadratic transformation $\varphi\colon x \to x^2 = c_0 + x_1 c_1 + (\frac{1}{3}x_2 + \frac{1}{6}x_1^2)c_2$ can be represented as acting on the coordinates to give

$$1\varphi = 1, \quad x_1\varphi = x_1, \quad x_2\varphi = \tfrac{1}{3}x_2 + \tfrac{1}{6}x_1^2.$$

The linearization of φ defines a set

$$M_2 = \{1, x_1, x_1^2, x_2\}$$

called the *linearizing set of monomial (coordinate) functions* with cardinality 4, and we write

$$\operatorname{Card} M_2 = 4.$$

M_2 then defines uniquely (up to isomorphism) an *induced vector space* B_2 of vectors

$$(u_0, u_1, u_{11}, u_2),$$

where the u_i's are coordinates with respect to a basis which we may write as

$$c_0, c_1, c_1 \otimes c_1, c_2.$$

Clearly $\dim B_2 = \operatorname{Card} M_2$. B_2 in this case is isomorphic to $\mathbf{R}^4$.

We shall call the map $R\colon A_2 \to B_2$,

$$x = c_0 + x_1 c_1 + x_2 c_2 \to (1, x_1, x_1^2, x_2),$$

which maps the plane of unit weight in A_2 onto a variety V in B_2, the *linearization map*.

$\tilde{\varphi}$ may be defined by linearity on the whole vector space B_2:

$$u_0\tilde{\varphi} = u_0, \quad u_1\tilde{\varphi} = u_1, \quad u_{11}\tilde{\varphi} = u_{11}, \quad u_2\tilde{\varphi} = \tfrac{1}{3}u_2 + \tfrac{1}{6}u_{11}.$$

By examining the structure of $\tilde{\varphi}$ in B_2 we can, in particular, deduce its action on the variety V, and by projecting back onto A_2 we can deduce the action of φ.

The representation of $\tilde{\varphi}$ as a matrix in this basis is given by the matrix A,

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{6} \\ 0 & 0 & 0 & \frac{1}{3} \end{bmatrix}.$$

It is sometimes convenient in studying $\tilde{\varphi}$ to look at its action on the basis vectors. This is given by the rows of A:

$$c_0\tilde{\varphi} = c_0, \quad c_1\tilde{\varphi} = c_1, \quad c_1 \otimes c_1\tilde{\varphi} = c_1 \otimes c_1 + \tfrac{1}{6}c_2, \quad c_2\tilde{\varphi} = \tfrac{1}{3}c_2.$$

REMARKS. 1. The linearizing functions are not unique. We could define, for instance, the function $t = x_1{}^2 - 4x_2$. Then

$$\begin{aligned} x_2\varphi &= \tfrac{1}{3}x_2 + \tfrac{1}{6}(t + 4x_2) = x_2 + \tfrac{1}{6}t, \\ t\tilde{\varphi} &= (x_1\varphi)^2 - 4(x_2\varphi) \\ &= x_1{}^2 - 4(x_2 + \tfrac{1}{6}t) = x_1{}^2 - 4x_2 - \tfrac{2}{3}t = \tfrac{1}{3}t. \end{aligned}$$

In order to obtain homogeneous linearizing functions we can take $x = x_0c_0 + x_1c_1 + x_2c_2$ where $x_0 = 1$, and then $t = x_1{}^2 - 4x_2x_0$. This polynomial function was the one used by Haldane working in the usual genetic basis and taking

$$x = x_0AA + x_1Aa + x_2aa, \quad \sum_{i=0}^{2} x_i = 1.$$

It is quite clear that in either basis we can always take our linearizing functions as homogeneous polynomials by multiplying suitably by either $x_0 = 1$ or $\sum x_i = 1$.

2. Clearly we are concerned with choosing a minimum number of linearizing functions in order to obtain a linearization of the problem. By a *linearization* we shall mean a *minimal linearization*.

3. It can sometimes happen that we can only obtain a canonical basis over the complex numbers, Heuch [**10**], in which case we obtain a complex induced vector space.

4. Linearization is very easily effected by working in a canonical basis. The linearizing functions which arise are monomials

$$x_1{}^{\alpha_1}x_2{}^{\alpha_2}\ldots x_n{}^{\alpha_n}.$$

We can call this a *canonical linearization.* Linearization in an arbitrary basis is usually difficult to obtain and will yield, in general, linearizing functions which are polynomials in $\mathbf{R}[x_1, x_2, \ldots, x_n]$ or $\mathbf{C}[x_1, x_2, \ldots, x_n]$, where these denote polynomial rings over the real and complex numbers respectively.

5. It is sometimes helpful to look at the transformation φ as defining a system of difference equations.

Writing $x_n = x_{n0}c_0 + x_{n1}c_1 + x_{n2}c_2$ to denote the population in the nth generation, we have

$$x_{n+1,0} = x_{n0}\varphi = x_{n0} = x_{00} = 1,$$

$$x_{n+1,1} = x_{n1}\varphi = x_{n1} = x_{01},$$

$$x_{n+1,2} = x_{n2}\varphi = \tfrac{1}{3}x_{n2} + \tfrac{1}{6}x_{n1}{}^2.$$

We can easily solve this explicitly by recursion. In this paper we are only concerned with obtaining a canonical linearization and describing the dimension of the induced vector space.

As Holgate [**11**] showed, we can, however, always take monomials to linearize quadratic transformations $x \to x^2$ in (Schafer) genetic algebras.

We shall now prove the main theorem of this section, namely, that the dimension of B_n is independent of the basis of A_n (not necessarily the canonical basis) and it is also independent of the linearizing functions and *a fortiori* of the particular construction used to obtain them. Thus we shall have shown the uniqueness of B_n (up to isomorphism) with respect to the particular quadratic transformation $x \to x^2$, where x is a general element of unit weight in A_n. However, the linearizing functions in the natural genetic basis will not have a simple structure and can always be taken as homogeneous polynomials in $n+1$ variables $x_0, x_1, x_2, \ldots, x_n$. In the proof we shall have to use the result that non-singular transformations of a basis in A_n induce non-singular transformations on the space of homogeneous polynomial functions. Thus, for example, different quadratic, cubic, quartic,... functions in $x_0, x_1, x_2, \ldots, x_n$ are mapped onto different quadratic, cubic, quartic,... functions respectively in $y_0, y_1, y_2, \ldots, y_n$ by a non-singular transformation. For this result we use a property of *Schläflian matrices* (also known as *induced matrices*). Finally, we shall see that although the linearizing functions are not unique, even in a particular basis, they are unique up to their degree. By that I mean that the same number of linearizing functions of the same degree are needed, whatever the basis in A_n, in order to linearize the quadratic transformation.

We remind the reader of some standard terminology using a simple example. Consider a two-dimensional vector space with coordinates (x_1, x_2) and a non-singular linear transformation $x = yA$, giving new coordinates (y_1, y_2), so that

$$x_1 = a_{11}y_1 + a_{21}y_2, \quad x_2 = a_{12}y_1 + a_{22}y_2.$$

The homogeneous quadratic form $Q = ax_1^2 + 2bx_1x_2 + cx_2^2$ will be transformed into $Q = a'y_1^2 + 2b'y_1y_2 + c'y_2^2$, where

$$(a', b', c') = (a, b, c)\begin{bmatrix} a_{11}^2 & a_{11}a_{21} & a_{21}^2 \\ 2a_{11}a_{12} & a_{11}a_{22} + a_{21}a_{12} & 2a_{21}a_{22} \\ a_{12}^2 & a_{12}a_{22} & a_{22}^2 \end{bmatrix}.$$

We say that A induces a transformation $\tilde{A}$: $(a, b, c) \to (a', b', c')$ defined by the above matrix, which is a *second-order Schläflian*, Muir [**15**].

In general for a homogeneous form f of degree r,

$$f = \sum_i a_i x_0^{\alpha_i} x_1^{\beta_i} x_2^{\gamma_i} \ldots x_n^{\nu_i},$$

where $\alpha_i + \beta_i + \ldots + \nu_i = r$ for all i, a linear transformation $(x_i) \to (y_i)$ induces a linear transformation of the coefficients of the form, given by $(a_i) \to (a'_i)$, which is given by a Schläflian matrix of the rth order.

By a result of Schläfli [**18**], quoted by Muir in [**15**], this matrix has a determinant equal to $|A|^\alpha$, where $\alpha = \binom{n+r-1}{n}$. Hence if A is non-singular, so is $\tilde{A}$.

This proves that different rth degree homogeneous forms are mapped onto different rth degree forms under the induced transformation of a change of basis in A_n. Hence the linearizing functions are unique up to their degree and the number of homogeneous linearizing functions of a given degree is an invariant of the algebra A_n with respect to a given quadratic transformation. If one chooses a minimal set of linearizing functions with respect to any basis then this will define uniquely the dimension of B_n.

Thus we have the following result.

THEOREM 1: the first fundamental theorem of genetic algebra. *The induced vector space corresponding to a linearization of a quadratic transformation is unique to within an isomorphism.*

COROLLARY. *The number of linearizing functions of a given degree is independent of the basis in A_n.*

3. The general Schafer genetic algebra

We consider an arbitrary (Schafer) genetic algebra (GA) A_n with the multiplication table defined as in § 1 and we shall assume that none of the constants of the algebra is zero. It is easily seen that in this case A_n is a special train algebra. This condition gives the most general genetic algebra A_n of a particular dimension in the sense that any other genetic algebra of the same dimension can only have at most the same number of non-zero structure constants as A_n. Letting some of these be zero decreases $\dim B_n$. Hence the general case gives us a maximal dimension for B_n for a particular quadratic transformation and for a general element of unit weight in a GA of dimension n.

Holgate [11] has given examples illustrating the linearization technique for A_1 and A_2. We illustrate the case for A_3, and state the results for A_4.

The algebra A_3

	c_0	c_1	c_2	c_3
c_0	c_0	$\lambda_{011}c_1+\lambda_{012}c_2+\lambda_{013}c_3$	$\lambda_{022}c_2+\lambda_{023}c_3$	$\lambda_{033}c_3$
c_1		$\lambda_{112}c_2+\lambda_{113}c_3$	$\lambda_{123}c_3$	0
c_2			$\lambda_{223}c_3$	0
c_3				0

$$x = c_0+x_1c_1+x_2c_2+x_3c_3,$$

$x\varphi = x^2$ represents the quadratic transformation.

The transformation φ may be viewed as acting on the coordinates $1, x_1, x_2, x_3$ as follows:

$$1\varphi = 1,$$

$$x_1\varphi = 2\lambda_{011}x_1,$$

$$x_2\varphi = 2\lambda_{012}x_1+2\lambda_{022}x_2+\lambda_{112}x_1^2,$$

$$x_3\varphi = 2\lambda_{013}x_1+2\lambda_{023}x_2+2\lambda_{033}x_3+\lambda_{113}x_1^2+2\lambda_{123}x_1x_2+\lambda_{223}x_2^2.$$

We define new variables $y_1 = x_1^2$, $y_2 = x_1x_2$, $y_3 = x_2^2$.

In the transition from x to $x\varphi$, x_ix_j is replaced by $x_i\varphi.x_j\varphi$, so for the induced transformation we define $(x_ix_j)\tilde{\varphi} = x_i\varphi.x_j\varphi$. Hence we have

$$y_1\tilde{\varphi} = 4\lambda_{011}^2y_1,$$

$$y_2\tilde{\varphi} = 2\lambda_{011}x_1(2\lambda_{012}x_1+2\lambda_{022}x_2+\lambda_{112}x_1^2),$$

$$y_3\tilde{\varphi} = (2\lambda_{012}x_1+2\lambda_{022}x_2+\lambda_{112}x_1^2)^2.$$

We now need to define further variables in order to linearize these equations:

$$y_4 = x_1^3, \quad y_5 = x_1^4, \quad y_6 = x_1^2 x_2.$$

Then

$$y_4\tilde{\varphi} = 8\lambda_{011}^3 y_4,$$

$$y_5\tilde{\varphi} = 16\lambda_{011}^4 y_5,$$

$$y_6\tilde{\varphi} = 4\lambda_{011}^2 x_1^2(2\lambda_{012}x_1 + 2\lambda_{022}x_2 + \lambda_{112}x_1^2).$$

The original quadratic transformation is now completely linearized. Altogether we needed six monomials to linearize the transformation. Hence B_3 is a 10-dimensional vector space induced by the set M_3 of coordinate functions,

$$M_3 = \{1, x_1, x_2, x_3, x_1^2, x_1x_2, x_2^2, x_1^3, x_1^4, x_1^2x_2\}.$$

Similarly for A_1 and A_2 we have the corresponding spaces B_1 and B_2 induced by $M_1 = \{1, x_1\}$ and $M_2 = \{1, x_1, x_1^2, x_2\}$ respectively. For A_4, B_4 is 36-dimensional and is induced by

$$M_4 = \{1, x_1, x_2, x_1^2, x_1^3, x_3, x_1x_2, x_4, x_2^2, x_1x_3, x_1^2x_2, x_1^4, x_1^5, x_2x_3, x_1x_2^2,$$
$$x_1^3x_2, x_1^2x_3, x_1^6, x_3^2, x_1x_2x_3, x_1^2x_2^2, x_1^3x_3, x_1^4x_2, x_1^7, x_1^5x_2,$$
$$x_1^3x_2^2, x_1^2x_2x_3, x_2^2x_3, x_1x_2^3, x_1^4x_3, x_1^8, x_1^6x_2, x_1^4x_2^2, x_1^2x_2^3, x_2^4, x_2^3\}.$$

It is convenient to define a polynomial function corresponding to each of these spaces:

$$P_1(1, x_1) = 1 + x_1,$$

$$P_2(1, x_1, x_2) = (1 + x_1)^2 + x_2,$$

$$P_3(1, x_1, x_2, x_3) = ((1 + x_1)^2 + x_2)^2 + x_3,$$

$$P_4(1, x_1, x_2, x_3, x_4) = (((1 + x_1)^2 + x_2)^2 + x_3)^2 + x_4.$$

By expanding these polynomials (from the outer brackets) one can see that the constituent monomials are given by the sets M_1, M_2, M_3, M_4 respectively. Thus we have produced explicitly a *generating polynomial* for the coördinate functions. We may call such a polynomial P a *generator* for M. By abuse of language we may say that P generates the vector space B. This polynomial plays a key role in obtaining the properties of B_n.

We shall write P_n for $P_n(1, x_1, \ldots, x_n)$.

THEOREM 2. *The linearizing set M_n of coordinate functions for the quadratic transformation $\varphi\colon x \to x^2$, where x is a general element of unit weight in A_n, is generated by the polynomial P_n defined recursively by $P_n = P_{n-1}^2 + x_n$, for $n = 1, 2, \ldots$, $P_0 = 1$.*

Proof. The theorem is clearly true for A_n, with $n \leqslant 4$.

Assume the theorem is true for A_{n-1} and suppose that B_{n-1} is induced by the set of monomials

$$M_{n-1} = \{1, y_1, y_2, y_3, y_4, y_5, z_1, y_6, z_2, z_3, y_7, \ldots\}$$

obtained by expanding P_{n-1} and relabelling the monomials by the variables y_i and z_i listed in a special (not quite uniquely determined) order as follows. The y_i's are either the linear or quadratic variables occurring in the transformation equations of A_{n-1}; they will be called *primary variables*. They are listed in the order in which they occur in the transformation equations. Thus $y_1 = x_1$, $y_2 = x_2$, $y_3 = x_1{}^2$, $y_4 = x_3$, $y_5 = x_1x_2, \ldots$, and we exclude any repetitions of such variables. The z_i's are further monomials generated by the quadratic primary variables in order to linearize φ. Thus $z_1 = x_1{}^3$ is generated by $\tilde{\varphi}$ acting on $y_5 = x_1x_2$. By abuse of language we shall say that x_1x_2 *generates* $x_1{}^3$. If z_i does not occur amongst any of the variables to its left, namely amongst the previously listed y's or z's, it is included in the listing; otherwise it is excluded. This avoids any repetitions. Similarly each z_i may generate further z's. These are listed in the ordering as they are generated, apart from repetitions. Thus any z_i will be generated by some y or z to the left of it. We call the z's *secondary variables*.

Consider A_n; $x_n\varphi$ contains a term in $x_{n-1}{}^2$ since all the λ's are assumed to be non-zero. Now $x_{n-1}\varphi$ is a linear combination of all the primary variables of M_{n-1} for the same reason, and

$$x_{n-1}{}^2\tilde{\varphi} = (x_{n-1}\varphi)^2.$$

Hence the induced transformation on $x_{n-1}{}^2$ will define further variables which are all the possible pairwise products (including squares) of the primary variables of M_{n-1}.

Suppose that the primary variables of M_{n-1} are $\{1, y_1, y_2, \ldots, y_p\}$. Hence y_iy_j are contained in M_n for all $i, j = 1, \ldots, p$.

We must show that all the other pairwise products (including squares) of the variables (primary and secondary) of M_{n-1} are included in M_n. There are two cases to consider.

Case 1: the variables $y_iz_j \in M_n$ for y_i, $z_j \in M_{n-1}$ for all i, j. We use induction on j. Consider y_iz_1. z_1 is generated by y_5, that is, $y_5\tilde{\varphi} = \alpha_1 z + \ldots$ and $y_iy_5 \in M_n$,

$$(y_iy_5)\tilde{\varphi} = (y_i)\tilde{\varphi}(y_5)\tilde{\varphi},$$

$$y_i\tilde{\varphi} = \beta y_i + \ldots.$$

Hence the induced transformation on y_iy_5 defines a variable y_iz_1. This therefore is included in M_n.

Now we make the inductive hypothesis that $y_i z_j \in M_n$ for fixed i and all $j < k$. Consider $y_i z_k$. By the ordering z_k is generated by either a y variable or a z variable to the left of it in the listing. If the former case holds

$$y_j\tilde{\varphi} = \gamma z_k + \ldots \quad \text{for some } j,$$

$$y_i\tilde{\varphi} = \beta y_i + \ldots .$$

The induced transformation on $y_i y_j$ will define $y_i z_k$. Hence $y_i z_k$ belongs to M_n. On the other hand, if z_k is generated by z_j, for $j < k$, $y_i z_j \in M_n$ by the inductive hypothesis and

$$(y_i)\tilde{\varphi} = \beta y_i + \ldots, \quad z_j\tilde{\varphi} = \delta z_k + \ldots .$$

Hence the induced transformation on $y_i z_j$ defines the variable $y_i z_k$. Therefore $y_i z_k$ belongs to M_n.

We conclude that $y_i z_j \in M_n$ for all j, fixed i. Since y_i was arbitrary, this is true for all i and j, including $y_0 = 1$.

Case 2: the variables $z_i z_j \in M_n$ for all z_i, z_j in M_{n-1}. By the ordering z_i is generated from $z_{i'}$ or $y_{i''}$ and z_j from $z_{j'}$ or $y_{j''}$. This gives four possibilities:

(a) z_i is generated from $y_{i''}$ and z_j from $y_{j''}$;

(b) z_i is generated from $z_{i'}$ $(i' < i)$ and z_j from $z_{j'}$ $(j' < j)$;

(c) z_i is generated from $y_{i''}$ and z_j from $z_{j'}$ $(j' < j)$;

(d) z_i is generated from $z_{i'}$ $(i' < i)$ and z_j from $y_{j''}$.

We examine each sub-case in turn.

Sub-case (a). We must show that $z_i z_j \in M_n$. Since $y_{i''}$ and $y_{j''}$ are primary, $y_{i''} y_{j''} \in M_n$,

$$(y_{i''} y_{j''})\tilde{\varphi} = (\alpha z_i + \ldots)(\beta z_j + \ldots).$$

Hence $z_i z_j \in M_n$.

Sub-case (b). Here we use double induction on i and j. For $i = j = 1$, $z_1^2 \in M_n$ since

$$y_5^2\tilde{\varphi} = (\alpha z_1 + \ldots)(\alpha z_1 + \ldots).$$

Assume that $z_{i'} z_{j'} \in M_n$ for all $i' < i$, $j' < j$. We shall show that this implies $z_i z_j \in M_n$ and hence by the second principle of induction (applied twice) $z_i z_j \in M_n$ for all z_i, z_j in M_{n-1} whenever Sub-case (b) is true. Since $z_{i'} z_{j'} \in M_n$ by the inductive hypothesis

$$(z_{i'} z_{j'})\tilde{\varphi} = (\alpha z_i + \ldots)(\beta z_j + \ldots)$$

defines $z_i z_j$. Hence $z_i z_j \in M_n$.

Sub-case (c). This follows easily by using Case 1, since $y_{i''} z_{j'} \in M_n$. Hence

$$(y_{i''} z_{j'})\tilde{\varphi} = (\alpha z_i + \ldots)(\beta z_j + \ldots),$$

which implies $z_i z_j \in M_n$.

Sub-case (d). The proof is similar to (c).

We have therefore shown that all pairwise products (including squares) of the variables (primary and secondary) of M_{n-1} are included in M_n.

Finally, we must show that the other primary variables in $x_n\varphi$ (excluding x_{n-1}^2) will occur in P_{n-1}^2. This is obvious since any such variable x_ix_j, with $i < n$, $j < n$, will occur as a product of linear primary variables in M_{n-1}. The secondary variables they generate will clearly be in P_{n-1}^2.

Hence the linearized space of A_n is given precisely by all the monomials in

$$P_n = P_{n-1}^2 + x_n.$$

Another characterization of the monomial functions is given by a linear diophantine inequality. This can be converted to a linear diophantine equality. Such equalities are important in the theory of partitions. We can also represent the monomials using *weight functions*.

If x_i has weight $w(x_i) = 2^{i-1}$, for $i = 1, 2, \ldots$, we define the weight of $x_1^{a_1}x_2^{a_2}\ldots x_n^{a_n}$ as $\sum_{i=1}^{n} a_i w(x_i)$.

The use of weight functions is simply a combinatorial device for investigating $\dim B_n$. There is no connection with the concept of baric weight.

PROPOSITION 1. *The monomials required to linearize the quadratic transformation $x \to x^2$, where x is a general element of unit weight in A_n, are all those monomials of weight less than or equal to 2^{n-1}. Equivalently $x_1^{a_1}x_2^{a_2}\ldots x_n^{a_n}$ is such a monomial if $a_1, a_2, \ldots, a_n$ are integral non-negative solutions of the inequality*

$$a_1 + 2a_2 + 2^2a_3 + 2^3a_4 + \ldots + 2^{n-1}a_n \leqslant 2^{n-1}, \quad \textit{for } n \geqslant 1.$$

Proof. Clearly the proposition is true for A_1. Consider

$$\{x_1^{a_1}x_2^{a_2}\colon a_1 + 2a_2 \leqslant 2,\ a_i \geqslant 0,\ a_i \in \mathrm{N}\}.$$

The solutions of the inequality are $\{(0,0), (0,1), (1,0), (2,0)\}$ which when substituted in $x_1^{a_1}x_2^{a_2}$ define the set M_2. Hence it is true for A_2.

Assume the truth of the proposition for A_{n-1}. Now $P_n = P_{n-1}^2 + x_n$ and by the inductive hypothesis P_{n-1} is the set of monomials m such that $w(m) \leqslant 2^{n-2}$ where $m = x_1^{a_1}\ldots x_{n-2}^{a_{n-2}}x_{n-1}^{a_{n-1}}$. Hence P_n consists of monomials m_im_j and x_n, where m_i, m_j are in M_{n-1}. Now

$$w(m_im_j) = w(m_i) + w(m_j) \leqslant 2^{n-2}.2 = 2^{n-1},$$

$$w(x_n) = 2^{n-1}.$$

Hence P_n consists of all the monomials m in $x_1, \ldots, x_n$ such that $w(m) \leqslant 2^{n-1}$, which establishes the truth of the proposition.

If $m = x_1^{a_1}x_2^{a_2}\dots x_n^{a_n}$, then

$$w(m) = a_1 + 2a_2 + 2^2a_3 + \dots + 2^{n-1}a_n \leqslant 2^{n-1}, \quad \text{for } n \geqslant 1.$$

Hence $a_1, \dots, a_n$ are non-negative integral solutions of this diophantine inequality.

PROPOSITION 2. *The number of monomials in M_n is precisely the number of integral, non-negative solutions of the linear diophantine equation*

$$a_1 + 2a_2 + 2^2a_3 + \dots + 2^n a_{n+1} = 2^n.$$

Proof. See the next section.

We can also define *generating functions* for the set M_n and for $u_n = \dim B_n = \operatorname{Card} M_n$. Define

$$\Phi(x;\, t) = (1 + x_1 t + x_1^2 t^2 + x_1^3 t^3 + \dots)(1 + x_2 t^2 + x_2^2 t^4 + \dots)$$
$$\times (1 + x_3 t^4 + x_3^2 t^8 + \dots)(1 + x_4 t^8 + \dots)\dots$$
$$= \prod_{i=1}^{\infty} (1 - x_i t^{2^{i-1}})^{-1},$$

where these are *formal power series*. Then clearly $\Phi(x;\, t)$ is a form of generating function for M_n in the sense that M_n will consist of all the coefficients of powers of t^i such that $i \leqslant 2^{n-1}$. We may obtain a *proper generating function* for M_n by considering $(1-t)^{-1}\Phi(x;\, t)$. Then the coefficient of $t^{2^{n-1}}$ in this expansion will give precisely the set M_n.

Similarly, by considering

$$(1-t)^{-1}\Phi(1;\, t) = (1-t)^{-1} \prod_{i=1}^{\infty} (1 - t^{2^{i-1}})^{-1},$$

we will have a *generating function* for u_n. Here u_n is the coefficient of $t^{2^{n-1}}$.

We have thus given various alternative characterizations of M_n and its cardinality, and hence of B_n and its dimension.

4. The dimension of the linearized space

A general problem in the theory of partitions is to partition a given number n with respect to some given quantities $b_1, b_2, \dots, b_m$. This is equivalent to solving the linear diophantine equation

$$y_1 b_1 + y_2 b_2 + \dots + y_m b_m = n \tag{1}$$

for integral, non-negative values $y_1, y_2, \dots, y_m$. The number of solutions of this equation is called the *denumerant* $D(m, n)$ (with respect to the *base* which is the fixed set of *summands* $b_1, b_2, \dots, b_m$). Sylvester was the first to investigate this problem and gave some very deep results using complex variable techniques. Several other writers, mentioned by Dickson in

[4, Chapter 3], have given some remarkable explicit formulae for the number of solutions of (1). However, they are usually impossible to evaluate for large m and n, and are of little practical value. E. T. Bell [1] has given a modern treatment of this problem. Formulae for $D(m,n)$ for special cases and small m and n can be obtained. A brief survey is given by Riordan in [16].

It is well known that if we define a generating function for $D(m,n)$ by

$$\Phi_m(t) = \sum_n D(m,n)t^n,$$

then

$$\Phi_m(t) = \{(1-t^{b_1})(1-t^{b_2})\dots(1-t^{b_m})\}^{-1}.$$

We define

$$D(m,0) = 1 \quad (m = 0,1,2,\dots),$$
$$D(0,n) = 0 \quad (n = 1,2,3,\dots).$$

Proof of Proposition 2. This is based on a recursive relation derived by Blom and Fröberg [3, p. 63].

Consider

$$\Phi_n(t) = \{(1-t)(1-t^2)\dots(1-t^{2^{n-1}})\}^{-1},$$
$$\Phi_{n+1}(t) = \{(1-t)(1-t^2)\dots(1-t^{2^n})\}^{-1}.$$

We have

$$\begin{aligned}\Phi_{n+1}(t) &= (1-t)^{-1}\Phi_n(t^2)\\ &= (1+t+t^2+t^3+\dots)(D(n,0)+t^2D(n,1)+t^{2^2}D(n,2)\\ &\qquad +\dots+t^{2^n-4}D(n,2^{n-1}-2)+t^{2^n-2}D(n,2^{n-1}-1)\\ &\qquad\qquad +t^{2^n}D(n,2^{n-1})+\dots).\end{aligned}$$

Equating coefficients of t^{2^n} in both sides, we obtain

$$D(n+1,2^n) = D(n,0)+D(n,1)+D(n,2)+\dots+D(n,2^{n-1}),$$

where $D(m,n)$ is with respect to the set of summands $1,2,2^2,\dots,2^{m-1}$. The left-hand side is the number of solutions of the equality

$$a_1+2a_2+2^2a_3+\dots+2^na_{n+1} = 2^n,$$

and the right-hand side is the number of solutions in the inequality of Proposition 1. This completes the proof.

Our interest centres on a very special case of this problem when the base consists of powers of 2, so that

$$b_1 = 1, \quad b_2 = 2, \quad b_3 = 2^2, \quad \dots, \quad b_m = 2^{m-1},$$

and $n = 2^{m-1}$.

Thus, from Proposition 2, $\dim B_n = D(n+1,2^n)$. In this case a more complete and easier recurrence solution can be given than in the general problem. A complete analysis of this case is given by Blom and Fröberg [3]. We shall also use a result of Mahler's [13] to give an asymptotic value for $u_n = \dim B_n$.

A number of recursive procedures are possible for computing u_n, depending on which method is used for characterizing B_n.

Method 1: *using the polynomials* P_n. The number of different monomials in the expansion of P_n will be denoted by u_n, and the number of different monomials in $P_n{}^m$ $(m = 0, 1, 2, \ldots)$ will be denoted by $u_{n,m}$.

We can easily derive a difference equation for $u_{n,m}$ by expanding

$$P_{n+1}{}^m = ((P_{n-1}{}^2 + x_n)^2 + x_{n+1})^m,$$

and forming the corresponding equation for $u_{n+1,m}$. Collecting terms together we eventually obtain

$$\begin{aligned} u_{n+1,m} = {} & [u_{n-1,4m} + u_{n-1,4m-2}] + 2[u_{n-1,4m-4} + u_{n-1,4m-6}] \\ & + 3[u_{n-1,4m-8} + u_{n-1,4m-10}] \\ & + \ldots + m[u_{n-1,4} + u_{n-1,2}] + m + 1 \quad \text{for } n \geqslant 2. \end{aligned}$$

One easily finds

$$u_{1,m} = m+1, \quad u_{2,m} = (m+1)^2, \quad u_{3,m} = 1 + \tfrac{1}{3}[4m^3 + 12m^2 + 11m].$$

Further explicit solutions for small values of n are possible in terms of Bernoulli polynomials. Using the recursion formula one can compute u_n.

For the general algebra A_n (where the λ's are non-zero) we obtain the following table:

	A_0	A_1	A_2	A_3	A_4	A_5	A_6	A_7
u_n	1	2	4	10	36	202	1828	27337

Method 2: *using a diophantine equality*. An alternative procedure is to use the characterization of B_n by the diophantine equality in Proposition 2. Blom and Fröberg [3] have given explicit details of the recursion procedure based on Bell's formula [1]. We give their result. Our sequence is

$$u_{m-1} = c_{m1} + 1, \quad \text{for } m = 1, 2, \ldots,$$

where

$$c_{m+1,i} = \sum_{j=i-1}^{2i} a_{ij} c_{mj} \quad (i, m \geqslant 1),$$

$$c_{ij} = 0 \quad (j \geqslant i),$$

$$c_{m0} = 1 \quad (\text{all } m),$$

and a_{ij} is given by either

$$a_{i+1,j+1} = 2a_{ij} + a_{i,j-1} \quad (i \geqslant 0, j \geqslant 0, a_{00} = 1)$$

or

$$a_{ij} = \binom{i}{j-i} 2^{2i-j} + \binom{i-1}{j-i+1} 2^{2i-j-2}.$$

They calculate first a table of the coefficients a_{ij} and then derive c_{mi}. Using this method we easily find $u_7 = 27337$, whereas the calculation is more protracted by the first method.

Mahler [**13**], in the course of a more general problem, investigated similar diophantine equations but with a base in terms of an arbitrary integer $a \geqslant 2$. Using his result, we obtain the asymptotic formula

$$\log u_n \sim \frac{(\log 2^n)^2}{2\log 2} = n^2 \log\sqrt{2} \quad (\text{as } n \to \infty).$$

5. A special case

Here we examine a class of (Schafer) genetic algebras A_n, with the usual canonical basis $c_0, c_1, \ldots, c_n$, which require only quadratic functions of the coordinates $x_1, \ldots, x_n$ to linearize quadratic transformations denoting random mating. This is related to a question raised by Moran [**14**, Chapter 2].

Proposition 3. *If A_n is a Schafer genetic algebra with idempotent c_0 such that $c_i c_j = \lambda_{ijn} c_n$, for $1 \leqslant i, j < n$, $\lambda_{ijn} \neq 0$, that is, the result of multiplication in the nil-ideal is a multiple of c_n, then A_n is an algebra which requires only quadratic functions to linearize a quadratic transformation $\varphi: x \to x^2$, where x is a general element of unit weight in A_n.*

Proof. The multiplication tables for A_2 and A_3 are:

	c_0	c_1	c_2
c_0	c_0	$\lambda_{011}c_1 + \lambda_{012}c_2$	$\lambda_{022}c_2$
c_1		$\lambda_{112}c_2$	0
c_2			0

	c_0	c_1	c_2	c_3
c_0	c_0	*	*	*
c_1		$\lambda_{113}c_3$	$\lambda_{123}c_3$	0
c_2			$\lambda_{223}c_3$	0
c_3				0

By direct computation we find that the corresponding linear spaces B_n are induced by the sets of coordinate functions

$$M_2 = \{1, x_1, x_2, x_1^2\},$$
$$M_3 = \{1, x_1, x_2, x_3, x_1x_2, x_2^2, x_1^2\},$$

respectively, and in each case $\dim B_n$ ($n = 2, 3$) is the maximum dimension for such a transformation of A_n; if any of the $\lambda_{ijk} = 0$ ($1 \leqslant i, j < n$) $\dim B_n$ would decrease.

Consider A_n with the above multiplication table. Now $x_n\varphi$ includes the maximum number of quadratic terms which can occur and the induced transformations on these will not introduce any other functions because of the linearity of $x_i\varphi$, for $i < n$.

REMARK. Any number of train roots may be zero. This will not alter B_n. The reason is that we are restricted to quadratic functions only.

If we extend the concept of a Bernstein algebra [**12**] we can define an *nth-order Bernstein algebra* as one in which equilibrium is reached after exactly n generations of panmixia:

$$x^{[n+2]} = x^{[n+1]}.$$

It is seen that the algebras in this section are all *second-order Bernstein algebras*, $x^{[4]} = x^{[3]}$, when all the train roots apart from 1 are zero.

The Bernstein algebras discussed in [**12**] would correspond to *first-order Bernstein algebras*.

The space B_n is generated by

$$(1+x_1+x_2+\ldots+x_{n-1})^2+x_n,$$

and

$$\dim B_n = \tfrac{1}{2}(n^2+n+2).$$

The following table gives the dimensions of the induced vector spaces for the algebras A_n:

	A_0	A_1	A_2	A_3	A_4	A_5	A_6	A_7
$\dim B_n$	1	2	4	7	11	16	22	29

6. Conclusions

Holgate's Theorem 2 [**11**] is an important theorem as it represents a major step forward in constructing a definitive and fundamental theory of genetic algebras, as we show in a subsequent paper. However, it is incomplete since the eigenvalues of the induced linear transformation explicitly depend on the constructed monomials which are not unique and are basis dependent, and the minimality of B_n is only shown with respect to a particular canonical basis. In this paper we prove that $\dim B_n$ is an invariant of A_n with respect to $\varphi\colon x \to x^2$; it then follows that the eigenvalues of $\tilde{\varphi}$ are also invariant from the theory in a subsequent paper where linearization is shown to be unique (up to similarity).

By considering the most general case of a genetic algebra we easily obtain some elegant generating polynomials in closed form. This is not generally the case for simpler algebras where many of the structure constants are zero. Methods of defining generating functions for M_n and u_n are described. The use of diophantine equalities enables us to examine the asymptotic behaviour of u_n.

The (Schafer) genetic algebras requiring only quadratic functions to linearize them have very simple structures. Again it is more convenient to consider the most general of such algebras.

Although we have assumed the existence of an idempotent (for simplicity), it is easily seen that this assumption can be dispensed with. Our results are clearly valid over the real and complex number fields.

Further, we see that the concept of linearization is a fundamental idea as the class of algebras in which quadratic transformations may be linearized is strictly larger than the class of Schafer algebras. Further, it is a technique which is clearly applicable in a wider context than in genetics and to more general non-linear transformations.

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